Quantum Morphisms

Lecture 13:

The Finale

Last Week

If $\mathcal{U}$ is the fundamental representation of $Qut(G)$, then the following are equivalence relations:

1) $x \sim_1 y$ if $u_{xy} \neq 0$;
2) $(x,x') \sim_2 (y,y')$ if $u_{xy} u_{x'y'} \neq 0$.

The equivalence classes are the orbits & orbitals of $Qut(G)$ respectively.

The orbits form an equitable partition
The orbitals form a coherent configuration.

$C_q^G(1,0) := \{\psi \in \mathbb{C}^{V(G)} : \mathcal{U}\psi = \psi \mathcal{U}^{\otimes 0}\}$ = span of char. vecs. of orbits.
$C_q^G(1,1) := \{M \in \mathbb{C}^{V(G) \times V(G)} : M\mathcal{U} = \mathcal{U}M\}$
= span of the characteristic matrices of the orbitals

This is a coherent algebra.

Thus if $(x,x'),(y,y') \in V(G) \times V(G)$ are in different classes of the coherent configuration of G, then $u_{xy} u_{x'y'} = 0$.

Theorem: Let G & H be graphs. Then $G \cong_{qc} H$ if & only if there is a QPM $\mathcal{P}$ s.t. $A_G \mathcal{P} = \mathcal{P} A_H$.

Theorem: Let G & H be connected graphs. Then $G \cong_{qc} H$ if & only if $\exists\, g \in V(G)$ & $h \in V(H)$ in the same orbit of $Qut(G \cup H)$.

Theorem: Let $\mathcal{A}_G$ & $\mathcal{A}_H$ be the quantum orbital algebras of G & H respectively. If $G \cong_{qc} H$, then there is an isomorphism (of coherent algebras) $\phi: \mathcal{A}_G \to \mathcal{A}_H$ s.t. $\phi(A_G) = A_H$.

Corollary: Let $\mathcal{A}_G$ & $\mathcal{A}_H$ be the coherent algebras of G & H respectively. If $G \cong_{qc} H$, then there is an isomorphism $\phi: \mathcal{A}_G \to \mathcal{A}_H$ s.t. $\phi(A_G) = \phi(A_H)$.

Corollary: If $G \cong_{qc} H$, then G and H are <u>not</u> distinguishable by the (2-dimensional) Weisfeiler-Leman algorithm.

Today: We give a diagrammatic description of the intertwiners of $Qut(G)$ in terms of bilabeled graphs. We will show that $G \cong_{qc} H$ if & only if G & H admit the same number of homomorphisms from any planar graph.

The latter is a quantum analog of a classical theorem of Lovász: $G \cong H$ if & only if G & H admit the same number of homomorphisms from any graph.

For a family of graphs $\mathcal{F}$, let $G \cong_{\mathcal{F}} H$ denote

$$hom(F, G) = hom(F, H) \text{ for all } F \in \mathcal{F}$$

where $hom(F,G) := \#$ homomorphisms from F to G.

The family $\mathcal{F}$	The relation $G \cong_{\mathcal{F}} H$
All graphs	Isomorphism [Lovász 1967]
Cycles	Cospectrality (of adjacency matrices)
Cycles & paths	Cospectral + cospectral complements
Trees	Fractional iso [Dvořák 2010; Dell, Grohe, Rattan 2018]
Treewidth $\leqslant k$	Indistinguishable by k-WL [Dvořák 2010; DGR 2018]
Treedepth $\leqslant k$	Ind. by FOL w/ counting of quantifier rank $\leqslant k$ [Grohe 2020]
Planar graphs	Quantum isomorphism

Recall the following theorem:

Theorem (Chassaniol):

$$C_q^G = \langle M^{1,2}, M^{1,0}, A_G \rangle_{+,\circ,\otimes,*}$$
$$C^G = \langle M^{1,2}, M^{1,0}, A_G, S \rangle_{+,\circ,\otimes,*}$$

$S(e_i \otimes e_j) = e_j \otimes e_i$ $\qquad (M^{\ell,k})_{i_1 \cdots i_\ell, j_1 \cdots j_k} = \begin{cases} 1 & \text{if } i_1 = \ldots = i_\ell = j_1 = \ldots = j_k \\ 0 & \end{cases}$

$$M^{0,0} = (|V(G)|)$$

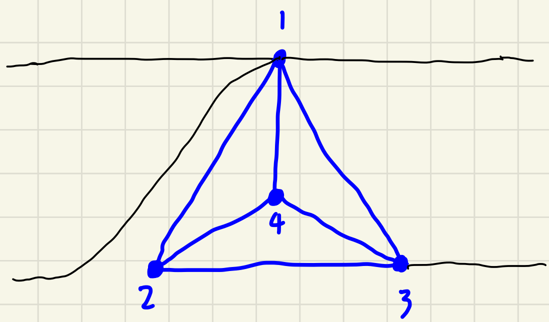
To obtain a diagrammatic description of the intertwiners of $Qut(G)$, i.e. of C_q^G, we first find such a description of $M^{1,2}$, $M^{1,0}$, & A_G, and then of the operations $\circ, \otimes, *$.

Bilabeled Graphs

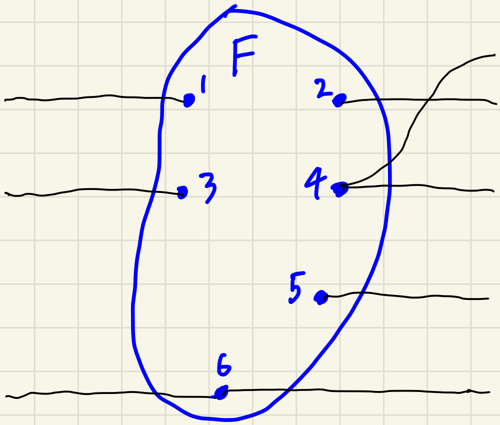
A bilabeled graph is a triple $\vec{F} = (F, \vec{a}, \vec{b})$ where F is a graph (loops allowed but no multiple edges) and $\vec{a} \in V(F)^{\ell}$, $b \in V(F)^{k}$ for some $\ell, k \geq 0$ are tuples of vertices of F (repeats allowed). We refer to $\vec{a}$ & $\vec{b}$ as the left and right (or output and input) tuples of $\vec{F}$. Sometimes: (ℓ, k)-bilabeled graph

Example: $\vec{F} = (K_4, (1,1), (1,3))$.

How do we draw this?



Abstract Example:



$\vec{F} = (F, (1,3,6), (4,2,4,5,6))$

Homomorphism Matrices

G a graph, $\vec{F} = (F, \vec{a}, \vec{b})$ an (ℓ, k)-bilabeled graph

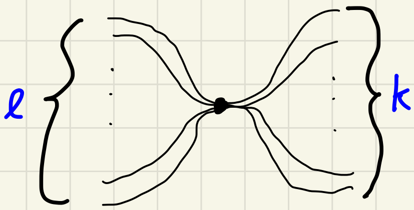
The G-*homomorphism matrix* of $\vec{F}$ is the $V(G)^\ell \times V(G)^k$ matrix $T^{\vec{F} \to G}$ where for $\vec{u} \in V(G)^\ell$, $\vec{v} \in V(G)^k$

$$(T^{\vec{F} \to G})_{\vec{u}, \vec{v}} = |\{\varphi \in Hom(F,G) : \varphi(\vec{a}) = \vec{u} \;\&\; \varphi(\vec{b}) = \vec{v}\}|$$

where $Hom(F,G)$ is the set of homomorphisms from F to G, and $\varphi(\vec{a}) = \vec{u}$ means $\varphi(a_i) = u_i \; \forall i = 1, \ldots, \ell$.

Examples:

$\vec{M}^{\ell,k} := (K_1, (\underbrace{1,\dots,1}_{\ell \text{ times}}), (\underbrace{1,\dots,1}_{k \text{ times}}))$ $\quad \ell + k \geq 1$



$$\left(T^{\vec{M}^{\ell,k} \to G}\right)_{\vec{u},\vec{v}} = \left|\{\varphi \in Hom(K_1, G) : \varphi(1) = u_i \;\&\; \varphi(1) = v_j \;\; \forall i,j\}\right|$$

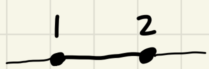
$$= \begin{cases} 1 & \text{if } u_1 = \dots = u_\ell = v_1 = \dots = v_k \\ 0 & \text{o.w.} \end{cases}$$

Thus $T^{\vec{M}^{\ell,k} \to G} = M^{\ell,k}$ for any G.

$\vec{M}^{0,0} := (K_1, \emptyset, \emptyset)$:

$T^{\vec{M}^{0,0} \to G} = (|V(G)|)$

$\vec{A} := (K_2, (1), (2))$:

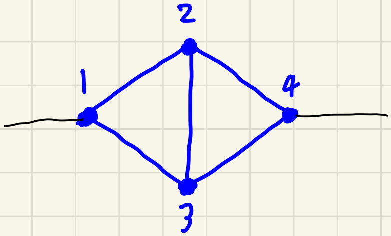


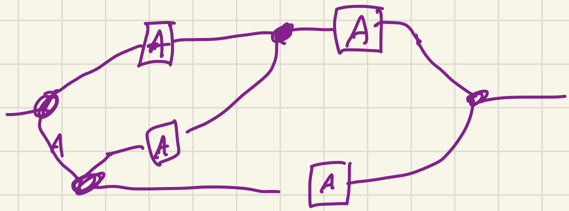
$$\left(T^{\vec{A} \to G}\right)_{u,v} = \left|\{\varphi \in Hom(K_2, G) : \varphi(1) = u \;\&\; \varphi(2) = v\}\right|$$

$$= \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{o.w.} \end{cases}$$

Thus $T^{\vec{A} \to G} = A_G$.

$\vec{F}$:



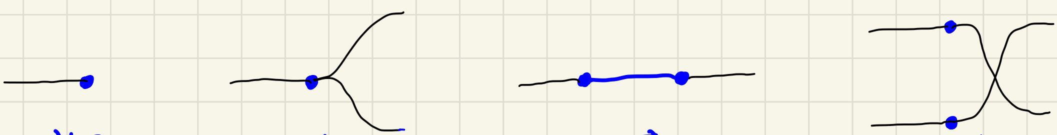


$$(T^{\vec{F}\to G})_{u,v} = |\{\varphi \in Hom(F,G) : \varphi(1)=u \ \& \ \varphi(4)=v\}|$$

$$= 2 \cdot \#\text{edges in } N(u) \cap N(v)$$

Remark: This matrix distinguishes the two orbits of non-edges in the Shrikhande graph.

Our Generators



$\vec{M}^{1,0}$ $\quad$ $\vec{M}^{1,2}$ $\quad$ $\vec{A}$ $\quad$ $\vec{S}$

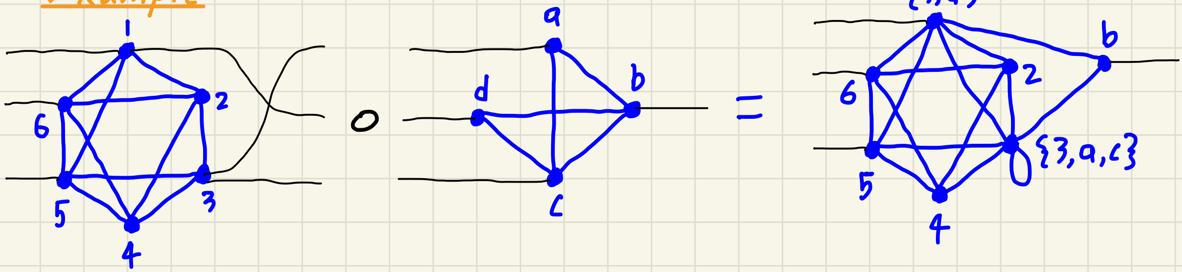
Operations with Bilabeled Graphs

Loops are kept, multiple edges are removed.

Composition/Product

Given an (ℓ,k)-bilabeled graph $\vec{F}=(F,\vec{a},\vec{b})$ and a (k,r)-bilabeled graph $\vec{K}=(K,\vec{c},\vec{d})$, their composition (or product), denoted $\vec{F}\circ\vec{K}$, is the (ℓ,r)-bilabeled graph $\vec{H}=(H,\vec{a}',\vec{d}')$ where H is the graph formed from $F \cup K$ by identifying the vertices b_j & c_j for all $j\in[k]$, and a_i' is the vertex of H that a_i became under these identifications, and similarly for d_i'.

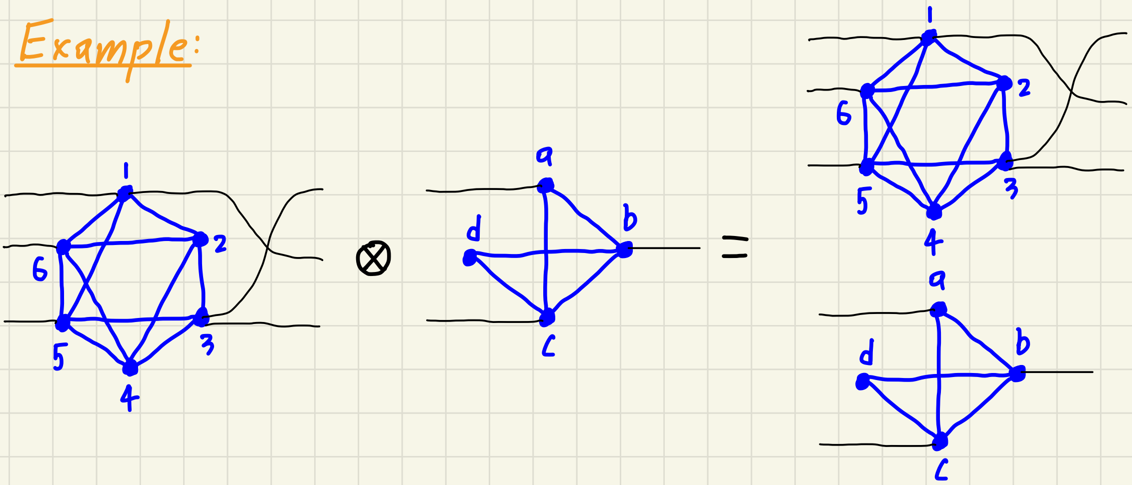
Example:



Tensor Product

Given an (ℓ, k)-bilabeled graph $\vec{F} = (F, \vec{a}, \vec{b})$ and a (r, s)-bilabeled graph $\vec{K} = (K, \vec{c}, \vec{d})$, their tensor product, denoted $\vec{F} \otimes \vec{K}$, is the $(\ell + r, k + s)$-bilabeled graph $(F \cup K, \vec{a}\vec{c}, \vec{b}\vec{d})$ where $\vec{a}\vec{c} = (a_1, \ldots, a_\ell, c_1, \ldots, c_r)$ is the concatenation of $\vec{a}$ + $\vec{c}$, and similarly for $\vec{b}\vec{d}$.

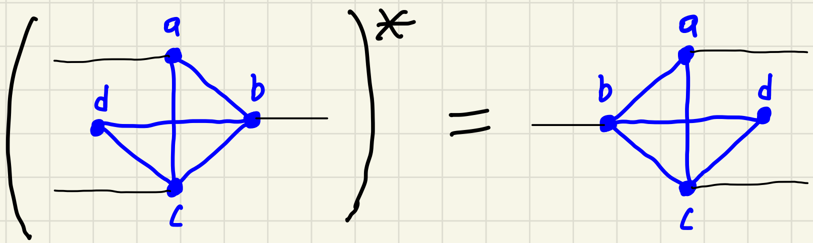
Example:



(Conjugate) Transpose

Given an (ℓ, k)-bilabeled graph $\vec{F} = (F, \vec{a}, \vec{b})$, its (conjugate) transpose, denoted $\vec{F}^*$, is the (k, ℓ)-bilabeled graph $(F, \vec{b}, \vec{a})$.

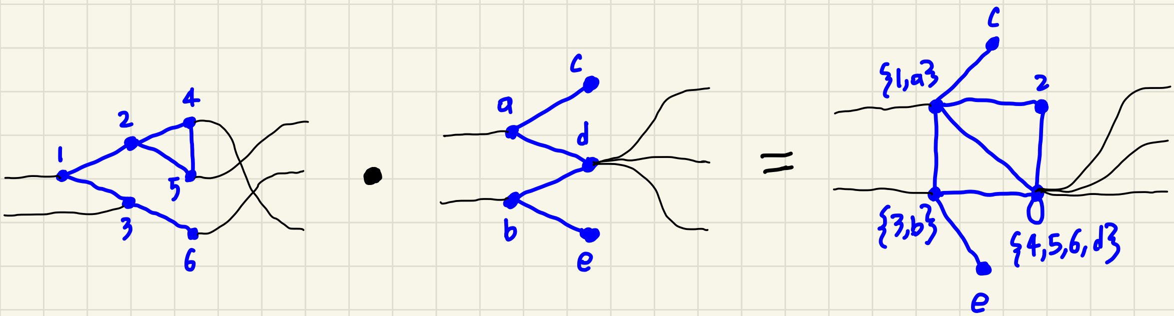
Example:



Schur Product

Given two (ℓ, k)-bilabeled graphs $\vec{F} = (F, \vec{a}, \vec{b})$ and $\vec{K} = (K, \vec{c}, \vec{d})$, their Schur product, denoted $\vec{F} \cdot \vec{K}$, is the (ℓ, k)-bilabeled graph $\vec{H} = (H, \vec{x}, \vec{y})$ where H is the graph formed from $F \cup K$ by identifying the vertices a_i & c_i $\forall i \in [\ell]$ and vertices b_j & d_j $\forall j \in [k]$, and where x_i is the vertex that a_i & c_i became, and y_j is the vertex b_j & d_j became.

Examples:



Theorem: Let $\vec{F} = (F, \vec{a}, \vec{b})$ be an (ℓ, k)-bilabeled graph and $\vec{K} = (K, \vec{c}, \vec{d})$ an (r, s)-bilabeled graph. For any graph G, the following hold:

1) if $k = r$, then $T^{\vec{F} \circ \vec{K} \to G} = T^{\vec{F} \to G} T^{\vec{K} \to G}$;
2) $T^{\vec{F} \otimes \vec{K} \to G} = T^{\vec{F} \to G} \otimes T^{\vec{K} \to G}$;
3) $T^{\vec{F}^* \to G} = (T^{\vec{F} \to G})^*$;
4) if $\ell = r$ & $k = s$, then $T^{\vec{F} \bullet \vec{K} \to G} = T^{\vec{F} \to G} \bullet T^{\vec{K} \to G}$.

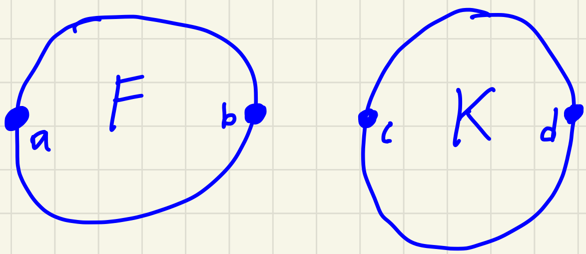
"Proof" for (1) when $\ell = k = r = 1$

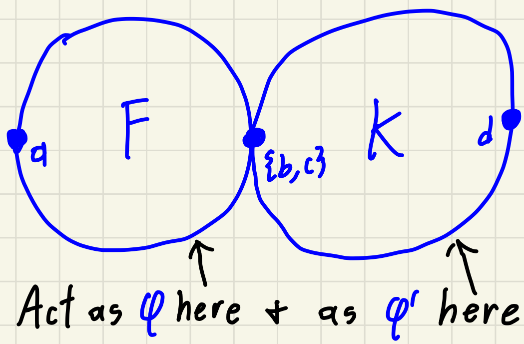
$\vec{F} = (F, (a), (b))$ & $\vec{K} = (K, (c), (d))$

Fix $u, v \in V(G)$. Then

$$\left(T^{\vec{F}\to G} T^{\vec{K}\to G}\right)_{uv} = \sum_{w \in V(G)} \left(T^{\vec{F}\to G}\right)_{uw} \left(T^{\vec{K}\to G}\right)_{wv}$$

$$= \sum_{w} \left|\{\varphi \in Hom(F, G) : \varphi(a) = u, \varphi(b) = w\}\right| \cdot \left|\{\varphi' \in Hom(K, G) : \varphi'(c) = w, \varphi'(d) = v\}\right|$$





$$= \left(T^{\vec{F} \circ \vec{K} \to G}\right)_{uv}.$$

Recall:

Theorem (Chassaniol):

$$\mathcal{C}_q^G = \langle M^{1,2}, M^{1,0}, A_G \rangle_{+,\circ,\otimes,*}$$
$$\mathcal{C}^G = \langle M^{1,2}, M^{1,0}, A_G, S \rangle_{+,\circ,\otimes,*}$$

Define the following:

$$\mathcal{C}_q := \langle \vec{M}^{1,2}, \vec{M}^{1,0}, \vec{A} \rangle_{\circ,\otimes,*}$$
$$\mathcal{C}_q(\ell,k) := \{ \vec{F} \in \mathcal{C}_q : \vec{F} \text{ has } \ell \text{ outputs \& } k \text{ inputs} \}$$

Then by the correspondence between bilabeled graph operations and matrix operations, we have that

$$\mathcal{C}_q^G(\ell,k) = \operatorname{span}\{ T^{\vec{F}\to G} : \vec{F} \in \mathcal{C}_q(\ell,k) \}.$$

Thus our goal is to describe the class $\mathcal{C}_q$.

Planar Bilabeled Graphs

Let $\vec{F} = (F, \vec{a}, \vec{b})$ be an (ℓ, k)-bilabeled graph. Define the graph $F^{\circ}(\vec{a}, \vec{b})$ to be the graph formed from the disjoint union of F and the $(\ell+k)$-cycle $\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_k, \ldots, \beta_1$ by adding edges $\alpha_i a_i \ \forall i \in [\ell]$ and $\beta_j b_j \ \forall j \in [k]$.

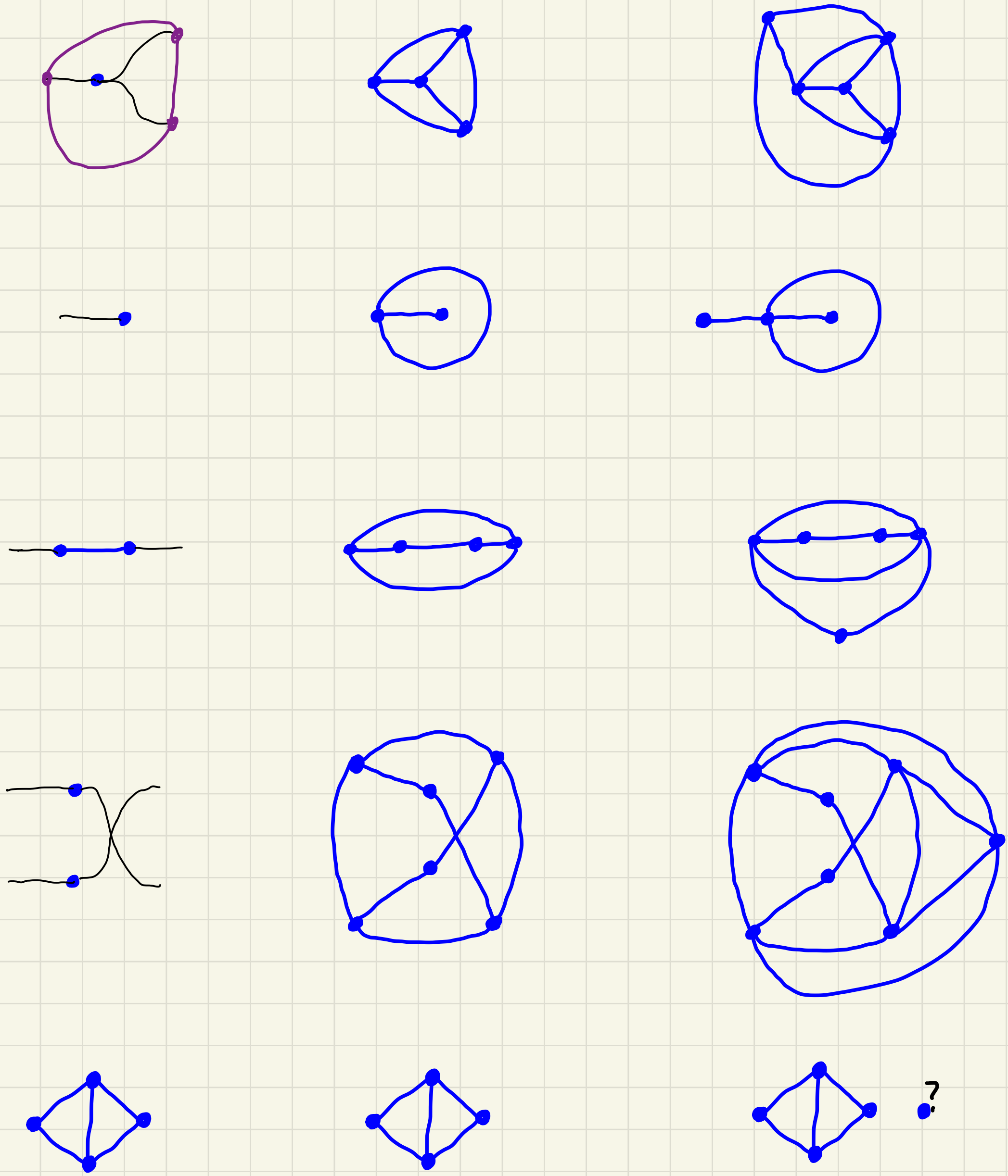
We refer to the cycle $\alpha_1, \ldots, \alpha_\ell, \beta_k, \ldots, \beta_1$ as the *enveloping cycle* of $F^{\circ}(\vec{a}, \vec{b})$.

Further define $F^{\odot}(\vec{a}, \vec{b})$ to be the graph obtained from $F^{\circ}(\vec{a}, \vec{b})$ by adding a new vertex adjacent to every vertex of its enveloping cycle.

We will typically just write F° and $F^{\odot}$.

Examples

$\vec{F}$ | F° | $F^{\odot}$



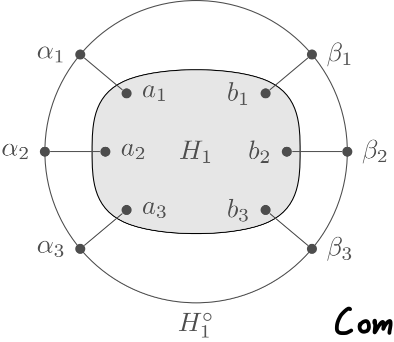
We say that $\vec{F} = (F, \vec{a}, \vec{b})$ is a planar bilabeled graph if the graph F° is planar. Equivalently, the graph F° has a planar embedding in which the enveloping cycle bounds the outer face.

We let $\mathcal{P}$ denote the set of planar bilabeled graphs and let $\mathcal{P}(\ell, k)$ denote the set of planar (ℓ, k)-bilabeled graphs.

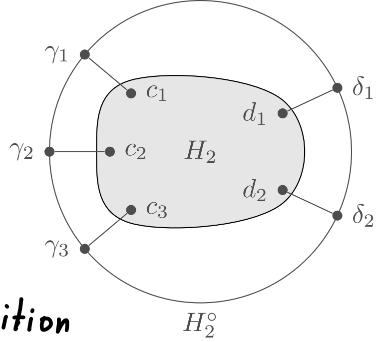
Lemma: If $\vec{F} = (F, \vec{a}, \vec{b})$ is an (ℓ, k)-bilabeled for $\ell + k \leq 1$, then $\vec{F} \in \mathcal{P}(\ell, k)$ if and only if the graph F is planar.

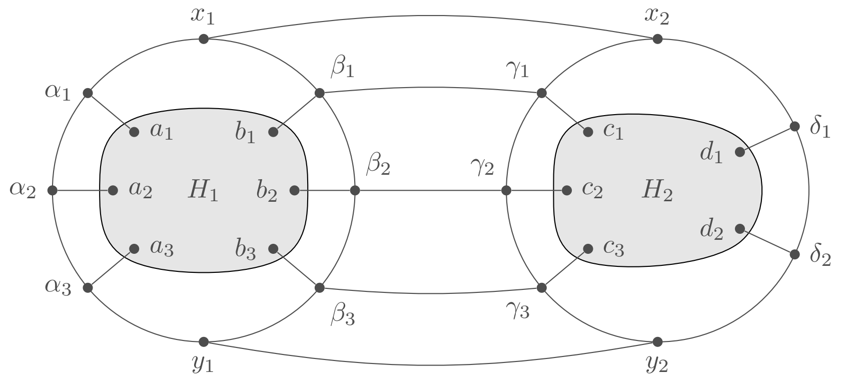
Theorem: $\mathcal{C}_q = \mathcal{P}$.

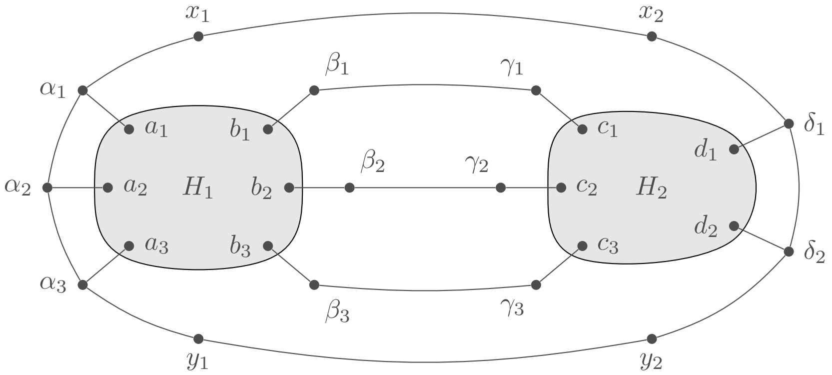
Proof $\mathcal{C}_q \subseteq \mathcal{P}$: The generators $\vec{M}^{1,2}, \vec{M}^{1,0}, \vec{A}$ of $\mathcal{C}_q$ are contained in $\mathcal{P}$. Thus we only need to show that $\mathcal{P}$ is closed under composition, tensor product, and transposition.



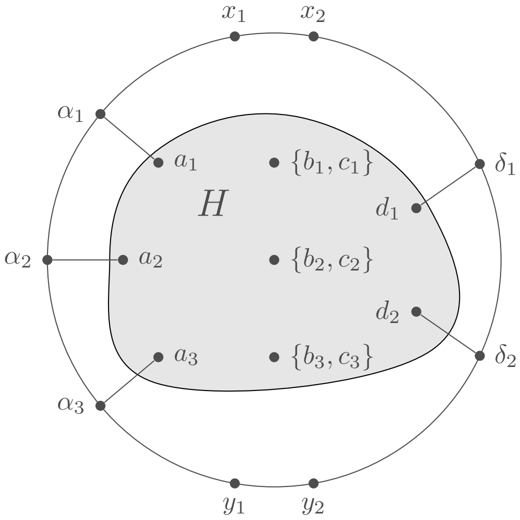
Composition



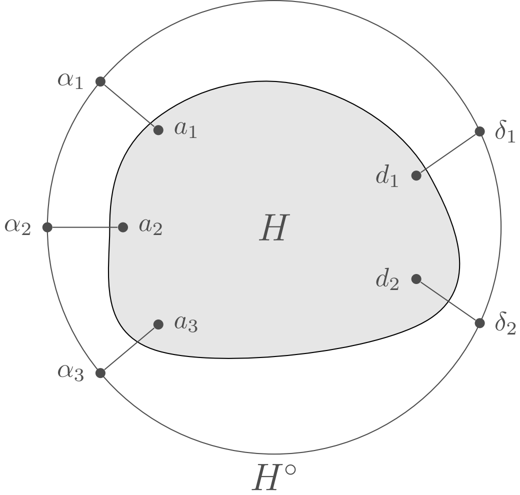

x_1
x_2
α_1
β_1
γ_1
δ_1
a_1
b_1
c_1
d_1
α_2
a_2
H_1
b_2
β_2
γ_2
c_2
H_2
a_3
b_3
c_3
d_2
δ_2
α_3
β_3
γ_3
y_1
y_2



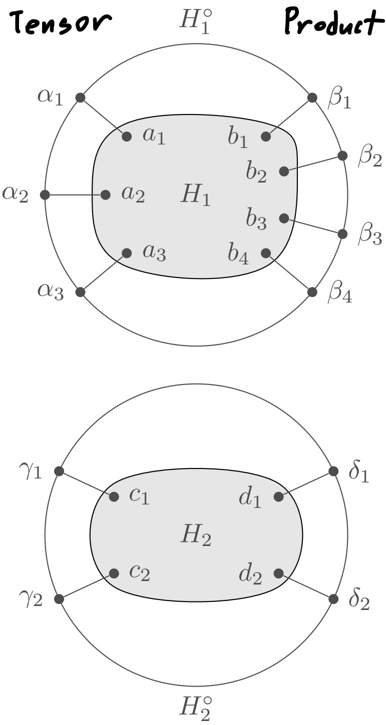
x_1
x_2
α_1
β_1
γ_1
δ_1
a_1
b_1
c_1
d_1
α_2
a_2
H_1
b_2
β_2
γ_2
c_2
H_2
a_3
b_3
c_3
d_2
δ_2
α_3
β_3
γ_3
y_1
y_2

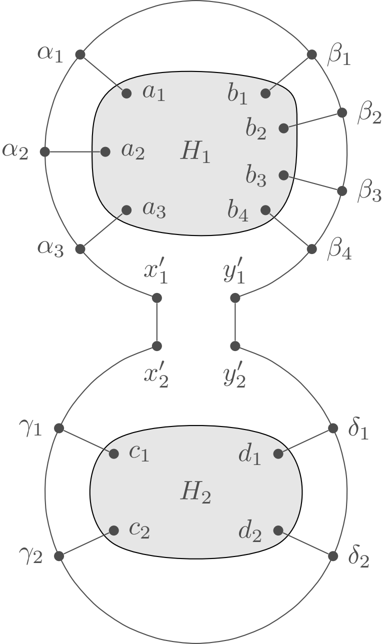


x_1
x_2
α_1
a_1
$\{b_1, c_1\}$
δ_1
H
d_1
α_2
a_2
$\{b_2, c_2\}$
d_2
a_3
$\{b_3, c_3\}$
δ_2
α_3
y_1
y_2

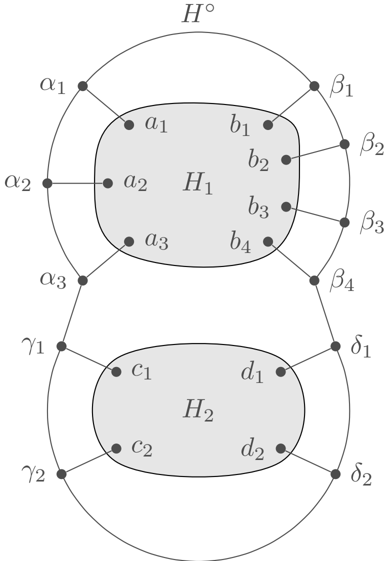

α_1
a_1
δ_1
d_1
α_2
a_2
H
d_2
a_3
δ_2
α_3
H°

Tensor Product





α_1
β_1
a_1
b_1
β_2
b_2
α_2
a_2
H_1
b_3
β_3
a_3
b_4
α_3
β_4
x'_1
y'_1
x'_2
y'_2
γ_1
δ_1
c_1
d_1
H_2
c_2
d_2
γ_2
δ_2

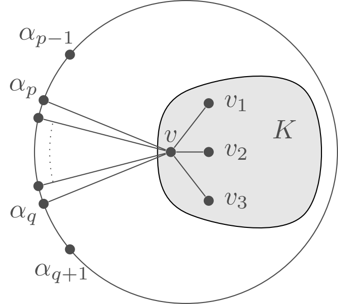


H°
α_1
a_1
b_1
β_1
β_2
b_2
α_2
a_2
H_1
b_3
β_3
a_3
b_4
α_3
β_4
γ_1
c_1
d_1
δ_1
H_2
c_2
d_2
γ_2
δ_2

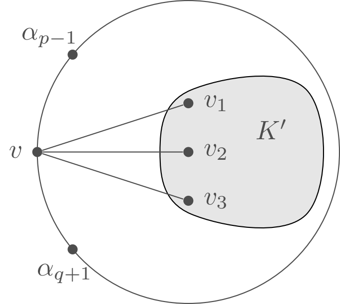
$\mathcal{P} \subseteq C_q$

We prove this by induction. The idea is to find a vertex we can carefully remove from $\vec{F} \in \mathcal{P}$ to obtain a smaller $\vec{F}' \in \mathcal{P}$. By induction $\vec{F}' \in C_q$, and then we show that $\vec{F}$ can be constructed from $\vec{F}'$ and some small bilabeled graphs from C_q.

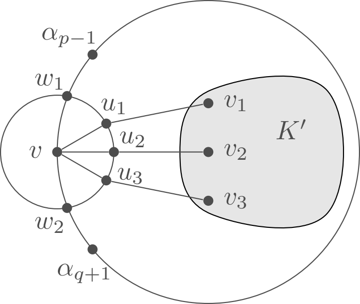
Lemma: Let $\vec{F} = (F, \vec{a}, \vec{b}) \in \mathcal{P}(\ell, k)$ with $\ell + k \geq 1$. Then there exists a vertex $v \in V(F)$ that occurs at least once in $\vec{a}$ or $\vec{b}$ such that its neighbors in the enveloping cycle of F° occur consecutively.

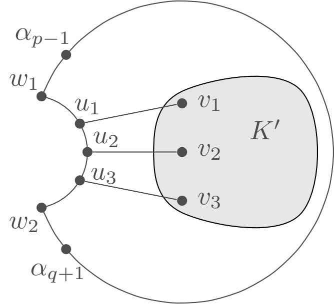


α_{p-1}
α_p
v_1
K
v
v_2
v_3
α_q
α_{q+1}

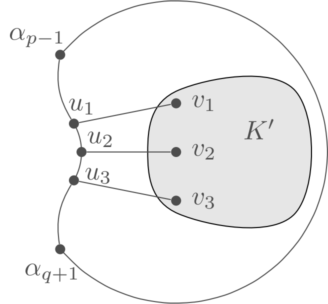


α_{p-1}
v_1
K'
v
v_2
v_3
α_{q+1}


α_{p-1}
w_1
u_1
v_1
K'
v
u_2
v_2
u_3
v_3
w_2
α_{q+1}

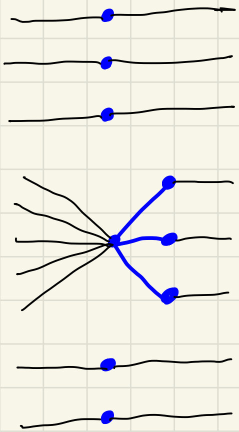


α_{p-1}
w_1
u_1
v_1
u_2
v_2
K'
u_3
v_3
w_2
α_{q+1}



α_{p-1}
u_1
v_1
K'
u_2
v_2
u_3
v_3
α_{q+1}

Compose this with:



to obtain the $\vec{F} \in \mathcal{P}$ we began with. □

$(F, (a), \emptyset)$

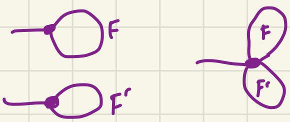
Let $hom(F_a, G_v)$ denote the number of homomorphisms from F to G that map a to v.

Corollary: Let G be a graph. Then $u, v \in V(G)$ are in the same orbit of $Qut(G)$ if and only if

$$hom(F_a, G_u) = hom(F_a, G_v)$$

for all connected planar graphs F and $a \in V(F)$.

Lemma: For $\ell + k \leq 2$, $\mathcal{P}(\ell, k)$ is closed under Schur product.



Theorem: Graphs G and H are quantum isomorphic if and only if $\hom(F,G)=\hom(F,H)$ for all planar graphs F.

Proof: ($\Rightarrow$) If $G \cong_{qc} H$, then there is a quantum permutation matrix $\mathcal{U}$ s.t. $\mathcal{U} A_G = A_H \mathcal{U}$. Also $\mathcal{U} M^{1,2} = M^{1,2} \mathcal{U}^{\otimes 2}$ & $\mathcal{U} M^{1,0} = M^{1,0} \mathcal{U}^{\otimes 0}$ for any quantum permutation matrix. From this we are able to show that if $T \in \langle M^{1,2}, M^{1,0}, A_G \rangle_{+,\circ,\otimes,*}$ is given by some expression in $M^{1,2}, M^{1,0}$, & A_G and $T' \in \langle M^{1,2}, M^{1,0}, A_H \rangle_{+,\circ,\otimes,*}$ is given by the same expression but with each occurrence of A_G replaced by A_H, then $\mathcal{U}^{\otimes \ell} T = T' \mathcal{U}^{\otimes k}$ for appropriate ℓ & k. By the correspondence between matrix and bilabeled graph operations, we obtain that $\mathcal{U}^{\otimes \ell} T^{\vec{F} \to G} = T^{\vec{F} \to H} \mathcal{U}^{\otimes k}$ for all $\vec{F} \in \mathcal{P}(\ell, k)$. Moreover, we can show that this correspondence is sum-preserving. Since $\operatorname{sum}(T^{\vec{F} \to G}) = \hom(F,G)$, we are done.

($\Leftarrow$) Suppose that $hom(F,G) = hom(F,H)$ for all planar graphs F. We may assume that G and H are both connected. We will show that there is an orbit of $Qut(G \cup H)$ intersecting both G and H.

By assumption $sum(T^{\vec{F}\to G}) = sum(T^{\vec{F}\to H})$ $\forall \vec{F} \in \mathcal{P}$. Since $\mathcal{P}(1,0)$ is closed under Schur product,

$sum((T^{\vec{F}\to G})^{\bullet m}) = sum(T^{\vec{F}^{\bullet m}\to G}) = sum(T^{\vec{F}^{\bullet m}\to H}) = sum((T^{\vec{F}\to H})^{\bullet m})$

$\forall \vec{F} \in \mathcal{P}(1,0)$. Thus $T^{\vec{F}\to G}$ + $T^{\vec{F}\to H}$ have the same multiset of entries $\forall \vec{F} \in \mathcal{P}(1,0)$, and this extends to linear combinations. A little more work shows that if $R = \sum_i \alpha_i T^{\vec{F}_i \to G}$ is the characteristic vector of an orbit of $Qut(G)$, then $R' = \sum_i \alpha_i T^{\vec{F}_i \to H}$ is the characteristic vector of an orbit of $Qut(H)$ of the same size. Pick $v \in V(G)$ and $v' \in V(H)$ in these orbits, i.e. such that $R_v = 1 = R'_{v'}$. Now let F be a connected planar graph and $a \in V(F)$. Thus $\vec{F} = (F,(a),\varnothing) \in \mathcal{P}(1,0)$.

Let $X = G \cup H$. We will show that $\hom(F_a, X_v) = \hom(F_a, X_{v'})$ thus proving v & v' are in the same orbit of $Qut(G \cup H)$.

Let $T = T^{\vec{F} \to G}$ & $T' = T^{\vec{F} \to H}$. Since F is connected, any homomorphism from F to X that maps a to v has its image contained in $V(G)$, and similarly for v' and $V(H)$. Thus

$$\hom(F_a, X_v) = \hom(F_a, G_v) = T_v$$

$$\hom(F_a, X_{v'}) = \hom(F_a, H_{v'}) = T'_{v'}$$

and so we want to show $T_v = T'_{v'}$. Since $T \in \mathcal{C}_q^G(1,0)$ and $T' \in \mathcal{C}_q^H(1,0)$, we have

$$R \bullet T = \alpha R$$

$$R' \bullet T' = \alpha' R'$$

where $\alpha = T_v$ & $\alpha' = T'_{v'}$. In fact $\alpha = \alpha'$ since $R \bullet T$ and $R' \bullet T'$ must have the same multiset of entries. Thus $v \in V(G)$ & $v' \in V(H)$ are in the same orbit of $Qut(G \cup H)$ and so $G \cong_{qc} H$. □